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PHASE AND ENVELOPE OF LINEAR FM PULSE-COMPRESSION

SIGNALS FROM HIGH-VELOCITY TARGETS

TECHNICAL DOCUMENTARY REPORT NO. ESD-TDR-64-128

NOVEMBER 1964

M. H. Ueberschaer

Prepared for

DIRECTORATE OF RADAR AND OPTICS

ELECTRONIC SYSTEMS DIVISION

AIR FORCE SYSTEMS COMMAND

UNITED STATES AIR FORCE

L. G. Hanscom Field, Bedford, Massachusetts

Project 750

Prepared by

THE MITRE CORPORATION
Bedford, Massachusetts
Contract AF 19 (628)-2390

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FOREWORD

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ABSTRACT

Equations for the phase and envelope of the output signal from a linear filter, matched to the transmitted signal, are derived. The transmitted signal is assumed to have a flat band-limited amplitude spectrum and a linear group delay. The input to the "matched filter" is the radar echo returned from a moving target whose velocity is essentially constant during the illumination time. It is shown that the returned signal is related to the transmitted signal by a time dilation. The resulting expressions for the phase and envelope are functions which involve Fresnel integrals. Approximations for these expressions are worked out. They are shown to be similar in form to those which are obtained when the returned signal is assumed to be related to the transmitted signal by a Doppler shift.

REVIEW AND APPROVAL

This technical documentary report has been reviewed and is approved.

HARRY BYRAM
Acting Chief, Radar Division
Directorate of Radar and Optics

PHASE AND ENVELOPE OF LINEAR FM PULSE-COMPRESSION SIGNALS FROM HIGH-VELOCITY TARGETS

SECTION 1
INTRODUCTION

This document is a first step in exploring the possibilities of using an "all pulse-compression" (linear FM) radar system to obtain accurate estimates of target range, radial velocity, and radial acceleration. The targets of primary interest here are high-velocity targets, such as artificial satellites.

It is well known that linear FM signals have an inherent coupling between range and velocity. However, it is also known that for targets whose range is varying slowly, and for small pulse-compression ratios, this can be overcome by transmitting alternately FM up and FM down, and taking the sum and difference of the resulting time-delays to obtain unambiguous estimates of target range and velocity, respectively. When the target range changes rapidly and the pulse-compression ratio is high, certain simplifying assumptions are no longer applicable, and the problem may become considerably more difficult.

This document investigates the problems from a fundamental point of view. The accuracies required in the ultimate estimation of target parameters dictate the necessity of measuring the phase of the target echo. We shall derive several expressions (with different degrees of exactness) for the phase and envelope of the output signal from a pulse-compression system. These expressions will be carefully compared and interpreted in a subsequent document.

We shall first review briefly the concepts of autocorrelation, matched filters, and correlation functions. Given an aperiodic time function f(t), such that

$$\int_{-\infty}^{\infty} f^2 (t) \, dt < \infty \ ,\tag{1}$$

we define the autocorrelation function of f(t) to be

$$\phi \, (\tau) = \int_{-\infty}^{\infty} f \, (t) \, f \, (t + \tau) \, dt \quad .\tag{2}$$

By a matched filter we mean a linear filter whose impulse response h(t) is a reflection of the time function to which it is matched, as illustrated in Fig. 1.

Input —→ f(t) | h(t) = f (T–t) | y(t) —→ Output
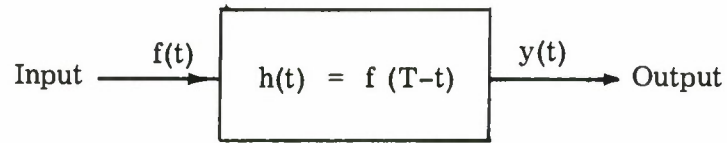
Fig. 1. Matched Filter

The quantity, T, is a constant which makes the filter realizable [i.e., we require that h(t) = 0 for t < 0]. The output is the convolution between the input and the impulse response, i.e.,

$$y(t) = \int_{-\infty}^{\infty} f \, (x) \, h \, (t-x) \, dx \ .\tag{3}$$

Thus,

$$y(t) = \int_{-\infty}^{\infty} f \, (x) \, f \, (x+T-t) \, dx \quad .\tag{4}$$

Letting $\tau = T-t$, we find that

$$y(t) = \int_{-\infty}^{\infty} f(x) \, f \, (x+\tau) \, dx \ ,\tag{5}$$

which is equivalent to $\phi \, (\tau)$ defined by Eq. (1).

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Thus, it is seen that matched filter reception and autocorrelation detection are identical processes, provided, of course, that the filter is truly "matched" to the input waveform.

In radar applications, the filter is often matched to the transmitted waveform. The target echo may be quite different from the transmitted signal. If this is the case, the output of the "matched filter" is no longer equal to the autocorrelation function but is now a cross-correlation function between the transmitted and received signals. It is often referred to loosely as an autocorrelation function. Regardless of its name, it is this function which we are interested in examining. We shall denote it by $y(t)$ throughout the rest of this report.

SECTION 2

PHYSICAL MODEL

Consider the model shown in Fig. 2 :
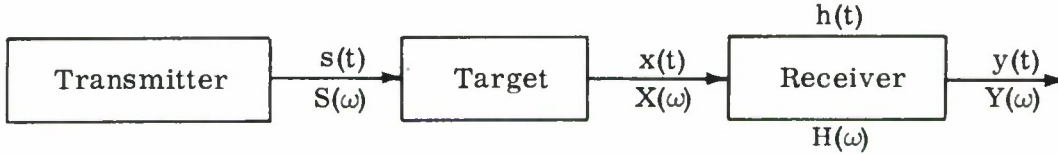
Fig. 2. Pulse-Compression Model

With the receiver matched to the transmitted signal, we have,

$$h(t) = s(T-t) \ , \quad H(\omega) = S^*(\omega) \exp[-j\omega T] \ , \tag{6}$$

where T makes the filter realizable, and the star denotes the complex conjugate.

A linear-FM rectangular pulse, lasting from time $t = 0$ to $t = T$, has the form

$$f(t) = \begin{cases} \cos\left[2\pi\left(f_1 + K \ \dfrac{t^2}{2}\right)\right]; & 0 \le t \le T \\[4mm] 0 & \text{elsewhere} \end{cases} \tag{7}$$

The Fourier transform of f(t) is a rather complicated function. However, it has been found* that the amplitude spectrum of f(t) becomes nearly rectangular as KT^2 (the time-bandwidth product) becomes large (>100). The present D-82 experimental radar facility employs a pulse-compression system with a time-bandwidth product of 1,000. A future system is proposed which

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*cf. Klauder, et al., "The Theory and Design of Chirp Radars," B.S.T.J., July 1960.

will have a time-bandwidth product of 10,000. Instead of synthesizing a simple function of time, the procedure here is to synthesize a simple function of frequency, denoted by $S(\omega)$.

The function $S(\omega)$ has approximately a rectangular amplitude spectrum $A(\omega)$, and a linear group delay* $T(\omega)$. For large time-bandwidth products, this corresponds approximately to a linear FM pulse.

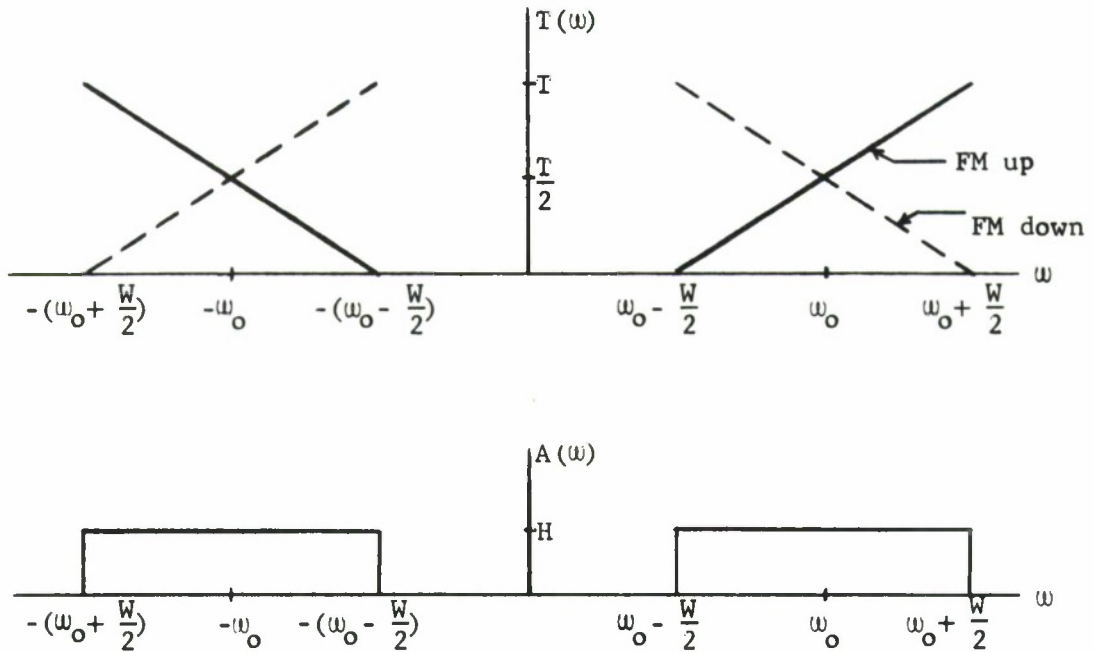For our analysis, we shall use the model shown in Fig. 3.

Fig. 3. Model of $A(\omega)$ and $T(\omega)$

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*Group delay is defined as

$$T(\omega) = - \frac{d\phi(\omega)}{d\omega} \quad,$$

where $\phi(\omega)$ is the phase of the spectrum.

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The respective equations are:

$$A(\omega) = \begin{cases} H \; ; \; \omega_0 - \dfrac{W}{2} \leq |\omega| \leq \omega_0 + \dfrac{W}{2} \\ \\ 0 \qquad\qquad\qquad \text{elsewhere} \end{cases} ; \qquad (8)$$

$$T(\omega) = \begin{cases} \dfrac{T}{W} |\omega| - \dfrac{T}{W} \left(\omega_0 - \dfrac{W}{2}\right) ; \; \omega_0 - \dfrac{W}{2} \leq |\omega| \leq \omega_0 + \dfrac{W}{2} \;\; \text{for FM up} \\ \\ -\dfrac{T}{W} |\omega| + \dfrac{T}{W} \left(\omega_0 + \dfrac{W}{2}\right) ; \; \omega_0 - \dfrac{W}{2} \leq |\omega| \leq \omega_0 + \dfrac{W}{2} \;\; \text{for FM down} \\ \\ 0 \qquad\qquad\qquad \text{elsewhere (for both FM up and FM down)} \end{cases}.$$

$$(9)$$

The spectrum $S(\omega)$ can be written as

$$S(\omega) = A(\omega)e^{j\phi(\omega)} , \qquad (10)$$

where $\phi(\omega) = - \displaystyle\int T(\omega) \, d\omega$ plus a constant of integration. Without loss of generality, we can let H = 1 in Fig. 3 and Eq. (8). Doing this, we obtain

$$S(\omega) = \begin{cases} e^{j\phi(\omega)} \; ; \;\; \omega_0 - \dfrac{W}{2} \leq |\omega| \leq \omega_0 + \dfrac{W}{2} \\ \\ 0 \qquad\qquad \text{elsewhere} \end{cases}. \qquad (11)$$

It is clear that $S(\omega)$ is an even function, since $A(\omega)$ and $T(\omega)$ are both even functions. For simplicity, we shall work only with the positive portion of the spectrum throughout the rest of this paper. It must be remembered that the actual spectra contain negative frequencies.

Integrating $T(\omega)$ and restricting ourselves to $\omega \geq 0$, we obtain

$$S(\omega) = \begin{cases} e^{j[c_0 + c_1 \omega + c_2 \omega^2]} & ; \quad \omega_0 - \dfrac{W}{2} \leq \omega \leq \omega_0 + \dfrac{W}{2} \\[4mm] 0 & \text{elsewhere} \end{cases} , \tag{12}$$

where

$$\begin{aligned} c_1 &= \frac{T}{W}\left(\omega_0 - \frac{W}{2}\right) & \left. \right\} \text{for FM up} \\[3mm] c_1 &= -\frac{T}{W}\left(\omega_0 + \frac{W}{2}\right) & \left. \right\} \text{for FM down} \end{aligned} \Bigg\} , \tag{13}$$

$$c_2 = \mp \frac{T}{2W} \begin{cases} - \text{ for FM up} \\[2mm] + \text{ for FM down} \end{cases} , \tag{14}$$

and c_0 is a constant of integration which will cancel out later. Equation (12) and Fig. 2 shall be applicable throughout this paper.

The following two pulse-compression systems are of particular interest:

Pulse-Compression Systems	T	$\dfrac{W}{2\pi}$	$\dfrac{\omega_0}{2\pi}$
No. 1 (10^3 system)	1 msec.	1 mc.	1280 mc.
No. 2 (10^4 system)	2 msec.	5 mc.	1280 mc.

The length of the transmitted pulse is approximately equal to T.

We shall regard the targets under discussion here as being essentially point targets. Let the maximum radial acceleration of our targets be 200 m/sec^2. Suppose a pulse of duration T is emitted at time $t = 0$. Let a given target be at range R when the leading edge of the pulse strikes it. Assuming

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free-space propagation, this occurs at the time $t = \dfrac{2R}{c}$, c being the velocity of light. Let the radial velocity and radial acceleration of the target at that time be equal to V and A, respectively.

During the time interval ($\approx$T) that the target is being illuminated, the change in radial velocity is

$$\Delta V = AT \leq 0.2 \text{ m/sec for system No. 1,}$$

$$\leq 0.4 \text{ m/sec for system No. 2,}$$

(since $A \leq 200$ m/sec^2). The corresponding change in range due to the acceleration term alone is

$$\Delta R = \frac{1}{2} AT^2 \leq 0.0001 \text{ m for system No. 1,}$$

$$\leq 0.0004 \text{ m for system No. 2,}$$

This is only a few degrees of phase shift for the 1280-megacycle radar, which is very small indeed. We are primarily interested in large radial velocities (of the order 10^3 and 10^4 m/sec, producing a $\Delta R = VT$ of the order of 1 to 20 meters).

Throughout this paper we shall regard the target velocity as being constant during the illumination time.

SECTION 3

DOPPLER-SHIFTED SIGNAL

One assumption which is often made is that the radar signal bounced off a moving target is a delayed and attenuated replica of the transmitted signal, except for a Doppler shift.* This is only an approximation. It is a fairly good one for small velocities. In Appendix I it is shown that the correlation function (under this assumption) is equal to

$$y(t) = \frac{2k}{\pi} \left(\frac{W'}{2} \right) \left(\frac{\sin \left[(t' + \beta) \frac{W'}{2} \right]}{\left[(t' + \beta) \frac{W'}{2} \right]} \right) \cos \left[(t' + \beta) \omega_0' + \delta \right] , \qquad (15)$$

The envelope has the familiar $\frac{\sin x}{x}$ form.

In Eq. (15)

$\quad$ k $\quad$ = some constant which depends on the radar cross section of the target,

$$W' = W - \left(\frac{2V}{c} \omega_0 \right) ,$$

$$t' = t - \frac{2R}{c} - T ,$$

$$\beta = \mp \frac{T}{W} \left(\frac{2V}{c} \omega_0 \right) \begin{bmatrix} - \text{ for FM up} \\ + \text{ for FM down} \end{bmatrix} ,$$

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*By this we mean that the spectrum of the returned signal X(ω) is related to the spectrum of the transmitted signal S(ω) by

$$X(\omega) = ke^{-j\omega \frac{2R}{c}} S \left(\omega + \frac{2V}{c} \omega_0 \right) ,$$

where $\frac{2V}{c} \omega_0$ is the "Doppler shift."

$$\omega_0' = \omega_0 - \frac{1}{2}\left(\frac{2V}{c}\ \omega_0\right),$$

$$\delta = \left(\frac{T}{W}\right)\left(\frac{2V}{c}\ \omega_0\right)\left[\pm\ \omega_0\left(1 - \frac{V}{c}\right) - \frac{W}{2}\right]\begin{bmatrix} + \text{ for FM up} \\ - \text{ for FM down} \end{bmatrix},$$

R = the target range when the pulse strikes it, and

V = the target radial velocity when the pulse strikes it.

The above expression was derived primarily for the purpose of comparing it with the more exact expressions to be developed in the next section.

SECTION 4

TIME-DILATED SIGNAL

In Appendix II it is shown that (under the assumption of essentially constant velocity during the illumination period) the relationship between the transmitted signal s(t) and the returned signal x(t) is given by

$$x(t) = k \, s \left(\frac{t - b}{a} \right) . \tag{17}$$

This corresponds to a "time dilation." Here

$$a = \frac{c+V}{c-V} \; ; \; b = \frac{2R}{c} , \tag{18}$$

and k, R, V, and c have the same meanings as before. In Appendix II we also obtain the Fourier transform of x(t):

$$X(\omega) = k \, a \, e^{-j\omega b} \, S(a\omega) . \tag{19}$$

The output of the receiver is then

$$Y(\omega) = X(\omega) \, S^*(\omega) \, e^{-j\omega T} = k \, a \, e^{-j\omega(b+T)} \, S(a\omega) \, S^*(\omega) . \tag{20}$$

From Eq. (10) it is clear that

$$S^*(\omega) = \begin{cases} e^{-j\phi(\omega)} & ; \; \omega_0 - \frac{W}{2} \leq \omega \leq \omega_0 + \frac{W}{2} \\ \\ 0 & \text{elsewhere} \end{cases} . \tag{21}$$

Similarly, we have

$$S(a\omega) = \begin{cases} e^{j\phi(a\omega)} & ; \; \omega_0 - \frac{W}{2} \leq a\omega \leq \omega_0 + \frac{W}{2} \\ \\ 0 & \text{elsewhere} \end{cases} . \tag{22}$$

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Since a > 0, this can be written as

$$S(a\omega) = \begin{cases} e^{j\phi(a\omega)} & ; \quad \dfrac{\omega_0 - \dfrac{W}{2}}{a} \le \omega \le \dfrac{\omega_0 + \dfrac{W}{2}}{a} \\ \\ 0 & \text{elsewhere} \end{cases} \qquad (23)$$

Combining (21) and (23), we obtain

$$S(a\omega)\, S^*(\omega) = \begin{cases} e^{j[\phi(a\omega) - \phi(\omega)]} & ; \quad \omega_0 - \dfrac{W}{2} \le \omega \le \dfrac{\omega_0 + \dfrac{W}{2}}{a} \\ \\ 0 & \text{elsewhere} \end{cases} \qquad (24)$$

Thus, over the non-zero interval, we have

$$\begin{aligned} \phi(a\omega) - \phi(\omega) &= c_0 + c_1\, a\omega + c_2\, (a\omega)^2 - [c_0 + c_1\, \omega + c_2\, \omega^2] \\ \\ &= c_1\, \omega\, (a-1) + c_2\, \omega^2\, (a^2 - 1) \end{aligned} \qquad (25)$$

Let

$$\begin{aligned} a - 1 &= \gamma = \dfrac{2V}{c-V} \\ \\ a^2 - 1 &= (a-1)(a+1) = \gamma\lambda \\ \\ \gamma &= \dfrac{2c}{c-V} \end{aligned} \qquad (26)$$

Substituting (24), (25) and (26) into (20), we obtain

$$Y(\omega) = \begin{cases} ka\ e^{-j\omega(b+T)}\ e^{j[\omega c_1 \gamma + \omega^2 c_2 \gamma\lambda]} & ; \quad \omega_0 - \dfrac{W}{2} \le \omega \le \dfrac{\omega_0 + \dfrac{W}{2}}{a} \\ \\ 0 & \text{elsewhere} \end{cases} \qquad (27)$$

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Let ω_0'' be the center of the non-zero region in the ω domain, and let W'' be the bandwidth of $Y(\omega)$; i.e., let

$$\omega_0'' - \frac{W''}{2} = \omega_0 - \frac{W}{2} \tag{28}$$

$$\omega_0'' + \frac{W''}{2} = \frac{1}{a}\left(\omega_0 + \frac{W}{2}\right) \tag{29}$$

Combining (28) and (29) with (18), and solving alternately for ω_0'' and then for $\frac{W''}{2}$ in terms of ω_0 and $\frac{W}{2}$, we obtain

$$\omega_0'' = \omega_0 \left(\frac{c}{c+V}\right) - \frac{W}{2}\left(\frac{V}{c+V}\right), \tag{30}$$

$$\frac{W''}{2} = \frac{W}{2}\left(\frac{c}{c+V}\right) - \omega_0\left(\frac{V}{c+V}\right). \tag{31}$$

These expressions are valid for a receding target. For an approaching target, we would obtain

$$\omega_0'' = \omega_0 \left(\frac{c}{c-V}\right) - \frac{W}{2}\left(\frac{V}{c-V}\right)$$

$$\frac{W''}{2} = \frac{W}{2}\left(\frac{c}{c-V}\right) - \omega_0\left(\frac{V}{c-V}\right)$$

The output spectrum can now be written as

$$Y(\omega) = \begin{cases} ka\, e^{j[\omega(-b-T+c_1 \gamma] + \omega^2 c_2 \gamma \lambda]} & ; \quad \omega_0'' - \frac{W''}{2} \le \omega \le \omega_0'' + \frac{W''}{2} \\ \\ 0 & \text{elsewhere} \end{cases} \tag{32}$$

We must remember that $Y(\omega)$ in Eq. (32) is only the positive half of the actual output spectrum. But since we have an even-frequency function, the output-time function $y(t)$ can be obtained by taking twice the real part of the Fourier transform of the positive half of the spectrum; i.e.,

$$y(t) = 2 \, \text{Re} \, \frac{ka}{2\pi} \int_{\left(\omega_0'' - \frac{W''}{2}\right)}^{\left(\omega_0'' + \frac{W''}{2}\right)} e^{j\omega[t - b - T + c_1 \gamma]} \, e^{j\omega^2 c_2 \gamma \lambda} \, d\omega \quad . \tag{33}$$

Here Re f(t) means "the real part of f(t)." Let

$$t' = t - b - T \tag{34}$$

$$= t - \frac{2R}{c} - T \quad ,$$

where $t' = 0$ corresponds to the peak of the autocorrelation function for the case of zero velocity. We then have

$$y(t) = 2 \, \text{Re} \, \frac{ka}{2\pi} \int_{\left(\omega_0'' - \frac{W}{2}\right)}^{\left(\omega_0'' + \frac{W}{2}\right)} e^{j\omega[t' + c_1 \gamma]} \, e^{j\omega^2 c_2 \gamma \lambda} \, d\omega \quad . \tag{35}$$

This expression is exact for a constant-velocity target, and very nearly exact if the percentage change in velocity over the pulse duration is small. However, it cannot be integrated in closed form. It is convenient to introduce the variable

$$\Omega = \omega - \omega_0'' \quad , \tag{36}$$

so that y(t) becomes

$$y(t) = 2\text{Re} \, \frac{ka}{2\pi} \, e^{j\sigma} \int_{-\frac{W''}{2}}^{\frac{W''}{2}} e^{j[c_3 \Omega + c_4 \Omega^2]} \, d\Omega \quad , \tag{37}$$

where

$$\sigma = (t' + \gamma c_1) \, \omega_0'' + \gamma \lambda c_2 \, (\omega_0'')^2 \quad ,$$

$$c_3 = t' + \gamma c_1 + 2\gamma \lambda c_2 \, \omega_0'' \quad , \quad \text{and}$$

$$c_4 = \gamma \lambda c_2 \quad . \tag{38}$$

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Equation (37) can be manipulated into the form

$$y(t) = 2 \operatorname{Re} k_1 \int_{u_2}^{u_1} e^{j\pi \frac{\alpha^2}{2}} dx$$

$$= 2 \operatorname{Re} k_1 \left[Z(u_1) - Z(u_2) \right] \qquad (39)$$

where

$$Z(x) = \int_0^x e^{j\pi \frac{\alpha^2}{2}} dx \qquad (40)$$

is the complex Fresnel integral. The resulting expression can then be put in terms of the simple Fresnel integrals,

$$C(x) = \int_0^x \cos\left(\pi \frac{\alpha^2}{2}\right) dx \quad ; \quad S(x) = \int_0^x \sin\left(\pi \frac{\alpha^2}{2}\right) dx \quad , \qquad (40a)$$

which are tabulated functions. This was done (the details are given in Appendix III), yielding

$$y(t) = \frac{x_0}{x_2} \left([C(x_3) - C(x_4)] \cos x_1 + [S(x_3) - S(x_4)] \sin x_1 \right) \quad , \qquad (41)$$

for FM up

$$y(t) = \frac{x_0}{x_2} \left([C(x_3) - C(x_4)] \cos x_1 - [S(x_3) - S(x_4)] \sin x_1 \right) \quad , \qquad (42)$$

for FM down

where

$$x_0 = \frac{ka}{\sqrt{2\pi}} \quad ,$$

$$x_1 = - \frac{(t')^2 + 2\gamma\lambda c_1 t'}{4\gamma\lambda c_2} \quad ,$$

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$$x_2 = \sqrt{\frac{\gamma \lambda T}{2W}} \quad ,$$

$$x_3 = \sqrt{\frac{\gamma \lambda T}{\pi W}} \left[\frac{c_3}{2 \gamma \lambda c_2} + \frac{W''}{2} \right] \quad , \text{ and}$$

$$x_4 = \sqrt{\frac{\gamma \lambda T}{\pi W}} \left[\frac{c_3}{2 \gamma \lambda c_2} - \frac{W''}{2} \right] \tag{43}$$

The other symbols are the same as before. We can also express y(t) in the following form:

$$y(t) = \left(\frac{x_0}{x_2} \right) R \cos (x_1 - \phi) \quad , \tag{44}$$

where

$$R = \sqrt{ \left[C(x_3) - C(x_4) \right]^2 + \left[S(x_3) - S(x_4) \right]^2 } \tag{45}$$

for both FM up and FM down, and

$$\phi = \left\{ \begin{array}{l} + \tan^{-1} \left[\dfrac{S(x_3) - S(x_4)}{C(x_3) - C(x_4)} \right] \text{ for FM up} \\[3em] - \tan^{-1} \left[\dfrac{S(x_3) - S(x_4)}{C(x_3) - C(x_4)} \right] \text{ for FM down} \end{array} \right\} . \tag{46}$$

These expressions are exact for a target whose velocity is constant over the pulse duration. Tables of the Fresnel integrals are available so that numerical answers of the correlation function y(t) can be obtained directly.

It would, however, be desirable to obtain approximate "closed-form" expressions for both the envelope R and the carrier term $\cos (x_1 - \phi)$. We are primarily interested in the representations which are valid in the vicinity of the peak of the envelope.

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In Appendix III we show that the peak occurs when $c_3 = 0$. This corresponds to the time

$$t = \frac{2R}{c} + T - \gamma c_1 - 2\gamma\lambda c_2 \omega_0'' \quad . \tag{47}$$

(Note that $\gamma = \frac{2V}{c-V} = 0$ when $V = 0$). Thus, we let $t'' = c_3$ in order to emphasize that the peak occurs at $t'' = 0$; i.e., let

$$t'' = t - \frac{2R}{c} - T + \gamma c_1 + 2\gamma\lambda c_2 \omega_0'' \quad . \tag{48}$$

In Appendix III it is also shown that (for the two pulse-compression systems we consider here) the arguments x_3 and x_4 of the Fresnel integrals are close to unity at the peak of the envelope (i.e., at $t'' = 0$) for large velocities. Thus, asymptotic approximations for R and ϕ which are valid in the immediate vicinity of the peak are unsuitable except for very low target velocities. However, the analysis of a simple Doppler-shifted signal [with Eq. (15) as the resulting ambiguity function] is expected to be adequate for low velocities. We shall, therefore, not attempt to get "closed-form" approximations for (45) and (46), which are valid in this region.

As we move away from the peak, the absolute values of x_3 and x_4 increase quite rapidly, allowing us to use asymptotic approximations of C(x) and S(x), for large x, yielding approximate expressions for R and ϕ which might provide some insight. This was done in Appendix III with the result

$$R = 2\Delta \left(\frac{\sin \pi A\Delta}{\pi A\Delta} \right) , \tag{49}$$

where

$$\left. \begin{aligned} A &= \sqrt{\frac{\gamma\lambda T}{\pi T}} \left(\frac{t''}{2\gamma\lambda c_2} \right) \\ \Delta &= \sqrt{\frac{\gamma\lambda T}{\pi W}} \left(\frac{W''}{2} \right) \end{aligned} \right\} , \text{ and} \tag{50}$$

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$$\phi = \pm \left(\frac{\gamma \lambda T}{2W}\right)\left[\left(\frac{t''}{2\gamma\lambda c_2}\right)^2 + \left(\frac{W''}{2}\right)^2\right] \begin{Bmatrix} + \text{ for FM up} \\ - \text{ for FM down} \end{Bmatrix} . \tag{51}$$

In Eq. (44) we expressed the correlation function by

$$y(t) = \left(\frac{x_0}{x_2}\right) R \cos (x_1 - \phi) .$$

This can be written as

$$y(t) = E \cos \theta . \tag{52}$$

In Appendix III the following approximate expressions for the envelope E and the phase θ were obtained:

$$E \approx a \left(\frac{2k}{\pi}\right)\left(\frac{W''}{2}\right)\left[\frac{\sin\left(t'' \frac{W''}{2}\right)}{\left(t'' \frac{W''}{2}\right)}\right] , \tag{53}$$

$$\theta \approx \omega_0'' t'' \pm \left(\frac{2V}{c-V}\right)\left(\frac{2c}{c-V}\right)\left(\frac{T}{2W}\right)\left[(\omega_0')^2 - \left(\frac{W'}{2}\right)^2\right]\begin{Bmatrix} + \text{ for FM up} \\ - \text{ for FM down} \end{Bmatrix} . \tag{54}$$

These approximations are good for

$$|t''| > 100 \, W'' \left(\frac{2V}{c-V}\right)\left(\frac{2c}{c-V}\right)\left(\frac{T}{2W}\right) \approx 100 \left(\frac{2V}{c}\right) T . \tag{55}$$

It is interesting to note that the $(t'')^2$ term in the expression for ϕ cancelled with $(t')^2$ term in the expression for x_1 [see Eq. (43)], so that $\theta = x_1 - \phi$ has only linear time dependence.

This is a direct result of the approximations we have made. The conclusion to be drawn is that the $(t'')^2$ term is negligibly small in the region indicated by Eq. (55). This may or may not be the case in the immediate vicinity of the peak.

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Let us consider a different method for approximating the output function. Going back to (37) we have

$$Y(t) = 2 \, Re \; \frac{ka}{2\pi} \; e^{j\sigma} \int_{-\frac{W''}{2}}^{\frac{W''}{2}} e^{j[c_3 \Omega + c_4 \Omega^2]} \, d\Omega \; .$$

It would be nice if we could simply ignore the Ω^2 term. Suppose we say that the answer we get by doing so is a good approximation if

$$|c_3 \Omega| > 100 \, |c_4 \Omega^2| \; . \tag{56}$$

Letting $t'' = c_3$ and substituting for c_4, we find that (56) becomes

$$|t'' \, \Omega| > 100 \left(\frac{2v}{c-V}\right)\left(\frac{2c}{c-V}\right)\left(\frac{T}{2W}\right) \Omega^2 \; .$$

Clearly, for any particular value of t'' this inequality is most difficult to satisfy when $|\Omega|$ is largest; i.e., when $\Omega = \pm \, \frac{W''}{2} \; .$

Thus, (56) is satisfied if

$$|t''| > 100 \left(\frac{2v}{c-V}\right)\left(\frac{2c}{c-V}\right)\left(\frac{T}{2W}\right) \frac{W''}{2} \approx 50 \left(\frac{2V}{c}\right) T \; . \tag{57}$$

When the Ω^2 term is ignored, (37) is integrated readily to yield, after taking the real part,

$$y(t) \approx a \left(\frac{2k}{\pi}\right)\left(\frac{W''}{2}\right) \frac{\sin\left[t'' \, \frac{W''}{2}\right]}{\left[t'' \, \frac{W''}{2}\right]} \cos \sigma \; . \tag{58}$$

We note that the envelope above is identical with Eq. (53). Let us compare the phase with Eq. (54). We have, from (38),

$$\sigma = (t' + \gamma c_1) \, \omega_0'' + \gamma \lambda c_2 \, (\omega_0'')^2 \; . \tag{38}$$

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When we express t' in terms of t'', with the aid of (48), we have

$$\omega_0'' \, t' = \omega_0'' \, t'' - \gamma \, c_1 \, \omega_0'' - 2 \gamma \lambda c_2 \, (\omega_0'')^2 \; . \tag{59}$$

Substituting (59) into (38), we obtain

$$\sigma = \omega_0'' \, t'' - \gamma \lambda c_2 \, (\omega_0'')^2 \; . \tag{60}$$

This becomes

$$\sigma = \omega_0'' \, t'' \pm \left(\frac{2v}{c-V}\right) \left(\frac{2c}{c-V}\right) \left(\frac{T}{2W}\right) (\omega_0'')^2 \quad \begin{array}{l} + \text{ for FM up} \\ - \text{ for FM down} \end{array} \tag{61}$$

Comparing this expression with the phase θ in Eq. (54) we find that the two expressions are the same except for the last factor which, in (54), is $\left[(\omega_0'')^2\right.$ $\left. - \left(\frac{W''}{2}\right)^2\right]$ instead of just $(\omega_0'')^2$. However, since $\omega_0'' \approx \omega_0$ (very closely) and $W'' < W$, and since ω_0 is about three orders of magnitude greater than W, we see that σ and θ are identical for all practical purposes.

Note that the region over which (58) is valid is, from (57),

$$|t''| > 50 \, W'' \, \left(\frac{2v}{c-V}\right) \left(\frac{2c}{c-V}\right) \left(\frac{T}{2W}\right) \approx 50 \left(\frac{2V}{c}\right) T \; .$$

In (55) we required $|t''|$ to be twice as large as that, which was a little more conservative.

It is quite remarkable that the simple way of approximating $y(t)$ shown above yields virtually the same result as the rather involved procedure of approximating the Fresnel integrals.

SUMMARY

We have derived three different expressions* for the output function $y(t)$. Each expression is of the form

$$y(t) = E \cos \theta .$$

(1) Simple Doppler Shift: $X(\omega) = k e^{-j\omega \frac{2R}{c}} \left[S \left(\omega + \frac{2V}{c} \omega_0 \right) \right] .$

$$
\left.
\begin{aligned}
E &= \frac{2k}{\pi} \frac{W'}{2} \frac{\sin\left[(t' + \beta) \frac{W'}{2}\right]}{\left[(t' + \beta) \frac{W'}{2}\right]} \\
\theta &= (t' + \beta) \omega_0' + \delta
\end{aligned}
\right\}
\qquad (62)
$$

where

$$\frac{W'}{2} = \frac{W}{2} - \frac{2V}{c} \omega_0 ,$$

$$t' = t - \frac{2R}{c} - T ,$$

$$\beta = \mp \frac{2V}{c} \omega_0 \frac{T}{W} \left.\begin{matrix} - \text{ for FM up} \\ + \text{ for FM down} \end{matrix}\right\} ,$$

$$\delta = \frac{2V}{c} \omega_0 \left[- \frac{T}{2} \pm \frac{T}{W} \omega_0 \left(1 - \frac{V}{c} \right) \right] \left\{\begin{matrix} + \text{ for FM up} \\ - \text{ for FM down} \end{matrix}\right\} ,$$

$$\omega_0' = \omega_0 \left(1 - \frac{V}{c} \right) ,$$

---

*Note: The expressions listed here may differ slightly in appearance from the corresponding ones in the test. They are identical, however.

$$c = \text{the speed of light,}$$

$$R = \text{the target's range when the pulse strikes it, and}$$

$$V = \text{the target's radial velocity when the pulse strikes it.}$$

(2) Time Dilation:
$$X(\omega) = k\, e^{-j\omega \frac{2R}{c}} \left[\frac{c+V}{c-V} \; S\left(\frac{c+V}{c-V}\, \omega \right) \right].$$

The exact expressions (within the limits of our assumptions) are:

$$E = \left(\frac{2k}{\pi} \right) \left(\frac{a}{2} \right) \sqrt{ \frac{\pi}{\gamma \lambda \left(\frac{T}{W} \right)} \left\{ [C(x_3) - C(x_4)]^2 + [S(x_3) - S(x_4)]^2 \right\} },$$

$$\theta = \pm \frac{\left[t'' \pm \gamma \lambda \left(\frac{T}{W} \right) \omega_0'' \right]^2}{2 \gamma \lambda \left(\frac{T}{W} \right)} \pm \tan^{-1} \left[\frac{S(x_3) - S(x_4)}{C(x_3) - C(x_4)} \right] \begin{cases} + \text{ for FM up} \\ - \text{ for FM down} \end{cases}. \tag{63}$$

The terms $C(x)$ and $S(x)$ are the Fresnel integrals, defined by Eq. (40a), and

$$a = \frac{c+V}{c-V} \;, \quad \gamma = \frac{2V}{c-V} \;, \quad \lambda = \frac{2c}{c-V} \;;$$

$$\omega_0'' = \omega_0 \left(\frac{c}{c+V} \right) - \frac{W}{2} \left(\frac{V}{c+V} \right),$$

$$\frac{W''}{2} = \frac{W}{2} \left(\frac{c}{c+V} \right) - \omega_0 \left(\frac{V}{c+V} \right),$$

$$t'' = t - \frac{2R}{c} - T \pm \gamma \left(\frac{T}{W} \right) \left[\omega_0 - \frac{W}{2} - \lambda\, \omega_0'' \right] \begin{cases} + \text{ for FM up} \\ - \text{ for FM down} \end{cases},$$

$$x_3 = \sqrt{\frac{\gamma \lambda}{\pi} \left(\frac{T}{W} \right)} \mp \left[\frac{t''}{\gamma \lambda \left(\frac{T}{W} \right)} + \frac{W''}{2} \right] \begin{cases} - \text{ for FM up} \\ + \text{ for FM down} \end{cases}, \text{ and}$$

$$x_4 = \sqrt{\frac{\gamma \lambda}{\pi} \left(\frac{T}{W} \right)} \mp \left[\frac{t''}{\gamma \lambda \left(\frac{T}{W} \right)} - \frac{W''}{2} \right] \begin{cases} - \text{ for FM up} \\ + \text{ for FM down} \end{cases}.$$

(3) Approximations for E and θ in Eq. (63).

$$E \approx \left(\frac{2k}{\pi}\right) a \left(\frac{W''}{2}\right) \left(\frac{\sin\left[t'' \ \frac{W''}{2}\right]}{\left[t'' \ \frac{W''}{2}\right]}\right) ;$$

$$\theta \approx \omega_0'' \ t'' \pm \gamma\lambda \left(\frac{T}{2W}\right)\left[\left(\omega_0''\right)^2 - \left(\frac{W''}{2}\right)^2\right] \begin{cases} + \text{ for FM up} \\ - \text{ for FM down} \end{cases} .$$

(64)

These approximations are expected to be quite good for

$$\left|t''\right| \gtrsim 50 \ W'' \ \gamma\lambda \ \left(\frac{T}{W}\right) \approx 100 \left(\frac{2V}{c}\right) T \ .$$

They may be adequate for considerably smaller values of $\left|t''\right|$.

The "exact" expressions in (63) are not easily interpreted. This will have to be done numerically. We did, however, determine that the peak of the envelope occurs at the time when $t'' = 0$. The peak of the approximate expression for the envelope in (64) occurs at exactly the same time. It is interesting, also, to note that the expressions in (64), for both the phase and the envelope, are virtually identical to those in (62) when the target velocity is small.

Due to the length of this document, we shall reserve the detailed interpretation, comparison, and application of the above expressions for a subsequent document, ESD-TDR-64-129.

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APPENDIX I

ANALYSIS OF A DOPPLER-SHIFTED SIGNAL

We assume that the relationship between the transmitted and received Fourier spectra is

$$X(\omega) = k\, e^{-j\,\frac{2R}{c}\,\omega}\, S(\omega - \omega_d) \ , \tag{I-1}$$

where

    k is the attenuation constant,

    R is the range of the target when the pulse strikes it, and

    ω_d is the "Doppler shift", given by

$$\omega_d = -\,\frac{2V}{c}\,\omega_0 \ , \tag{I-2}$$

where V is the radial velocity of the target when the pulse strikes it. (We are here using the convention that V is positive when the target is receding from the radar.) Consider the case of a receding target, and let

$$\alpha = -\,\omega_d = \frac{2V}{c}\,\omega_0 \ . \tag{I-3}$$

If Eq. (11) is substituted for the transmitted spectrum, (I-1) becomes:

$$X(\omega) = \begin{cases} k\, e^{-j\omega\,\frac{2R}{c}}\, e^{j\phi(\omega + \alpha)} & ; \quad \omega_0 - \dfrac{W}{2} \le \omega + \alpha \le \omega_0 + \dfrac{W}{2} \\[2em] 0 & \text{elsewhere} \end{cases} \tag{I-4}$$

Again, we must remember that this is only the positive half of the frequency spectrum. Equation (I-1) can be written as

$$X(\omega) = \begin{cases} k\, e^{-j\omega \frac{2R}{c}} \; e^{j\phi(\omega + \alpha)} & ; \quad \omega_0 - \alpha - \dfrac{W}{2} \le \omega \le \omega_0 - \alpha + \dfrac{W}{2} \\ \\ 0 & \text{elsewhere} \end{cases} \tag{I-5}$$

We have

$$Y(\omega) = X(\omega)\, S^*(\omega)\, e^{-j\omega T} . \tag{I-6}$$

By use of Eq. (I-5), this becomes

$$Y(\omega) = \begin{cases} k\, e^{-j\omega \left(\frac{2R}{c} + T\right)} \; e^{j[\phi(\omega + \alpha) - \phi(\omega)]} & ; \quad \omega_0 - \dfrac{W}{2} \le \omega \le \omega_0 - \alpha + \dfrac{W}{2} \\ \\ 0 & \text{elsewhere} \end{cases}$$

$$\tag{I-7}$$

Now,

$$\phi(\omega) = C_0 + C_1\, \omega + c_1\, \omega^2 , \tag{I-8}$$

with c_1 and c_2 given by (13) and (14). Thus,

$$\phi(\omega + \alpha) - \phi(\omega) = c_0 + c_1\,(\omega + \alpha) + c_2\left(\omega^2 + 2\omega\alpha + \alpha^2\right) - \left[\, c_0 + c_1\,\omega + c_2\,\omega^2\right]$$

$$= \omega\left(2\,c_2\,\alpha\right) + \left(c_1\,\alpha + c_2\,\alpha^2\right)$$

$$\tag{I-9}$$

Let

$$2\,c_2\,\alpha = \beta , \quad c_1\,\alpha + c_2\,\alpha^2 = \delta . \tag{I-10}$$

Further, let ω_0' be the center of the non-zero region of $Y(\omega)$, and let W' be the bandwidth of $Y(\omega)$; i.e., let

$$\omega_0' = \omega_0 - \frac{\alpha}{2} \quad , \quad \frac{W'}{2} = \frac{W-\alpha}{2} \quad . \tag{I-11}$$

Then we obtain

$$Y(\omega) = \begin{cases} k\, e^{j\delta}\, e^{-j\omega\left(\frac{2R}{c} + T\right)}\, e^{j\omega\beta} & ; \quad \omega_0' - \frac{W}{2} \le \omega \le \omega_0' + \frac{W'}{2} \\[2em] 0 & \text{elsewhere} \end{cases} \tag{I-12}$$

Our desired time function $y(t)$ (i.e., the correlation function) is twice the real part of the inverse Fourier transform of $Y(\omega)$; i.e.,

$$y(t) = 2\mathrm{Re}\ \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega)\, e^{j\omega t}\, d\omega \quad . \tag{I-13}$$

Substituting (I-12), we have

$$y(t) = 2\mathrm{Re}\ \frac{1}{2\pi} \int_{\omega_0' - \frac{W'}{2}}^{\omega_0' + \frac{W'}{2}} k\, e^{j\delta}\, e^{j\omega\left[t - \frac{2R}{c} - T + \beta\right]}\, d\omega \tag{I-14}$$

Let

$$t' = t - \frac{2R}{c} - T \quad . \tag{I-15}$$

We can integrate $y(t)$ directly, obtaining

$$y(t) = \frac{k}{\pi}\ \mathrm{Re}\ e^{j\delta}\ \left[\frac{e^{j\omega\,[t' + \beta]}}{j\,[t' + \beta]}\right]_{\omega_0' - \frac{W'}{2}}^{\omega_0' - \frac{W'}{2}}$$

$$= \frac{k}{\pi}\ \mathrm{Re}\ e^{j\delta}\, e^{j[t' + \beta]\, \omega_0'} \left(\frac{e^{j[t' + \beta]\frac{W''}{2}} - e^{-j[t' + \beta]\frac{W'}{2}}}{j\,[t' + \beta]}\right), \tag{I-16}$$

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from which we finally obtain the desired expression

$$y(t) = \frac{2k}{\pi} \left(\frac{W'}{2}\right) \left(\frac{\sin\left[(t' + \beta)\,\dfrac{W'}{2}\right]}{\left[(t' + \beta)\,\dfrac{W'}{2}\right]}\right) \cos\left[(t' + \beta)\,\omega_0 + \delta\right] \quad . \qquad \text{(I-17)}$$

APPENDIX II

RELATIONSHIP BETWEEN TRANSMITTED AND RECEIVED RADAR SIGNALS

For simplicity, we shall consider a point target with constant cross section. Suppose the transmitter sends an impulse in the direction of the target at time t = 0. The transmitted signal is given by

$$s_1(t) = \delta(t) , \tag{II-1}$$

where $\delta(t)$ is the delta function. Assume free-space propagation. Let the target be at a distance

$$R_1 = ct_1 \tag{II-2}$$

when the impulse hits it, c being the velocity of light. The returned echo will then also be an impulse which will strike the antenna at time $2t_1$, i.e.,

$$x_1(t) = k \delta(t - 2t_1) = k \delta\left(t - \frac{2R}{c}\right) , \tag{II-3}$$

where the constant k depends on the radar cross section of the target. Let another impulse be transmitted a short time τ later, so that

$$s_2(t) = \delta(t - \tau) . \tag{II-4}$$

For $t_1 > \tau$, this impulse will be at a position in space equal to $R_1 - c\tau$ when the first impulse strikes the target. Let the target have radial velocity V and radial acceleration A at time t_1. For V << c, this second impulse will strike the target at approximately an interval τ after the first impulse did. We have shown in the text that for the targets and the radar parameters of interest to us, the acceleration has negligible effect (over the illumination time) on the velocity and range. Here we assume τ to be shorter than the total illumination time.

Thus, the relative velocity between the second impulse and the target at time t_1 is $\Delta V = c - V$. The distance between the target and this impulse is $\Delta R = c\tau$. The time required for this impulse to "catch up" with the target is

$$\Delta t = \frac{\Delta R}{\Delta V} = \frac{c\tau}{c-V} \ .$$

(II-5)

The target will be at range

$$R_2 = R_1 + V\Delta t = R_1 + \frac{Vc}{c-V}\,\tau$$

(II-6)

when this impulse strikes it. The echo returned from this impulse is then given by

$$x_2(t) = k\,\delta\left(t - \tau - \frac{2R_2}{c}\right) = k\,\delta\left(t - \tau - \frac{2R_1}{c} - \frac{2V}{c-V}\,\tau\right).$$

(II-7)

Thus, the two returned pulses are a time $\tau\left(1 + \frac{2V}{c-V}\right)$ apart, while the two transmitted pulses are only a time τ apart. A time-dilation has taken place.

Since we can think of our actual transmitted signal as consisting of a sequence of impulses, we see that the target may be regarded as a time-varying ideal delay line in cascade with a time-invariant attenuator. This is illustrated in Fig. II-1.
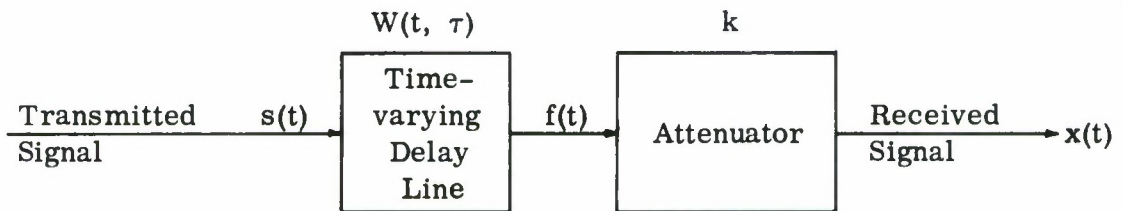

Fig. II-1. Ideal Delay Line

The delay line is a linear, time-varying filter. The output $y(t)$ of such a filter is related to the input $x(t)$ by*

*cf. Laning and Battin "Random Processes in Automatic Control," McGraw-Hill, 1956, p. 226.

$$f(t) = \int_{-\infty}^{t} s(\tau)\, W(t,\tau)\, d\tau \ , \qquad\qquad (\text{II-8})$$

where $W(t,\tau)$ is the time-varying impulse response. We may think of it as a curve in the $t - \tau$ plane which, for the case of constant velocity, is a straight line, as indicated in Fig. II-2.
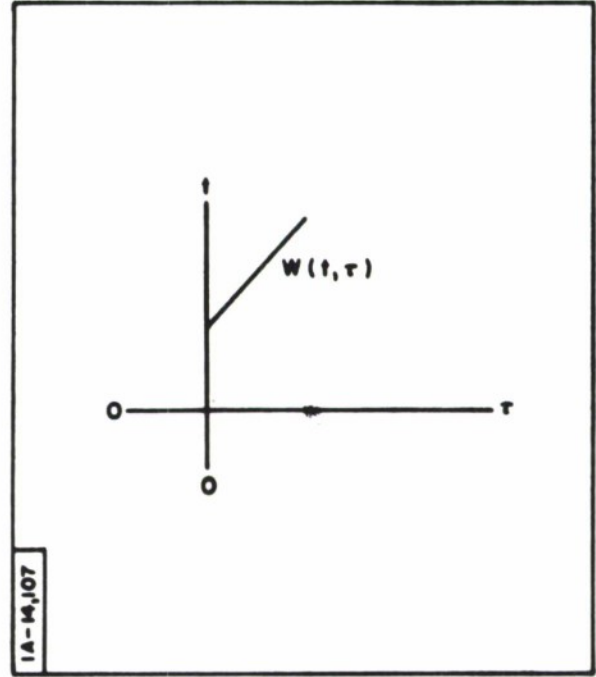
Fig. II-2. Constant Velocity Case

Here τ is the " input time" and t is the "response time." From Eq. (II-7), it is apparent that the equation of this line is given by

$$W(t,\tau) = \delta\left(t - \frac{2R_1}{c} - \tau\left[1 + \frac{2V}{c-V} \right] \right). \qquad\qquad (\text{II-9})$$

For convenience let $R = R_1$. Recognizing that $W(t,\tau) = 0$ for $t < \tau$, we find that (II-9) becomes

$$f(t) = \int_{-\infty}^{\infty} s(\tau)\, \delta\left(t - \frac{2R}{c} - \tau\left[1 + \frac{2V}{c-V} \right] \right) dt = s\left(\frac{t - \dfrac{2R}{c}}{1 + \dfrac{2V}{c-V}} \right). \quad (\text{II-10})$$

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But, $1 + \dfrac{2V}{c-V} = \dfrac{c+V}{c-V}$. Letting

$$b = \frac{2R}{c} \quad \text{and} \quad a = \frac{c+V}{c-V} \ , \tag{II-11}$$

we obtain

$$f(t) = s\left(\frac{t-b}{a}\right) \tag{II-12}$$

The Fourier transform of f(t) is given by

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} s\, \frac{t-b}{a}\, e^{-j\omega t}\, dt \tag{II-13}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} s(u)\, e^{-j\omega b}\, e^{-j\omega(au)}\, du$$

$$= a\, e^{-j\omega b}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} s(u)\, e^{-j(a\omega)u}\, du \ .$$

We recognize the last line of (II-13) as equal to

$$F(\omega) = e^{-j\omega b}\, a\, S(a\omega) \ . \tag{II-14}$$

Thus, the signal returned from the target is related to the transmitted signal s(t) by

$$x(t) = k\, s\left(\frac{t-b}{a}\right) , \tag{II-15}$$

and its Fourier transform $X(\omega)$ is related to $S(\omega)$ by

$$X(\omega) = k\, a\, e^{-j\omega b}\, S(a\omega) \ . \tag{II-16}$$

Equations (II-15) and (II-16) are physically plausible, as we shall now show. Suppose we send two impulses, one at $t = 0$ and one at $t = \tau$; i.e., let

$$s(t) = \delta(t) + \delta(t - \tau) \ . \tag{II-17}$$

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Then

$$x(t) = k \left[\delta\left(\frac{t-b}{a}\right) + \delta\left(\frac{t-b}{a} - \tau\right) \right]$$

$$= k \left[\delta\left(t - \frac{2R}{c}\right) + \delta\left(t - \frac{2R}{c} - \frac{c+V}{c-V}\tau\right) \right] . \qquad \text{(II-18)}$$

Thus, the received signal is attenuated, delayed and stretched (since the target is receding).

Next consider a CW signal; i.e., let

$$S(\omega) = \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \qquad \text{(II-19)}$$

from (II-16) we obtain, for the received signal,

$$X(\omega) = ka\, e^{-j\omega b} \left[\delta\, (a\omega - \omega_0) + \delta\, (a\omega + \omega_0) \right]$$

$$= ka\, e^{-j\omega b} \left[\delta\left(\omega - \frac{\omega_0}{a}\right) + \delta\left(\omega + \frac{\omega_0}{a}\right) \right] \qquad \text{(II-20)}$$

The new carrier frequency is therefore equal to $\dfrac{\omega_0}{a}$, where $a = \dfrac{c+V}{c-V}$.

Now,

$$\frac{1}{a} = \frac{1}{\frac{c+V}{c-V}} = \frac{1}{1 + \frac{2V}{c-V}} \qquad \text{(II-21)}$$

for $V < \dfrac{c}{3}$ this can be expanded in a power series, i.e.,

$$\frac{1}{a} = 1 - \left(\frac{2V}{c-V}\right) + \left(\frac{2V}{c-V}\right)^2 - \left(\frac{2V}{c-V}\right)^3 + \ldots \qquad \text{(II-22)}$$

For $V \ll c$ this becomes approximately equal to $1 - \dfrac{2V}{c}$, so that the new frequency is

$$\frac{\omega_0}{a} \approx \omega_0 \left(1 - \frac{2V}{c}\right) \qquad \text{(II-23)}$$

which we recognize as the familiar "doppler-shifted" frequency.

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APPENDIX III

ANALYSIS OF A TIME-DILATED SIGNAL AND
APPROXIMATIONS TO THE FRESNEL INTEGRAL EXPRESSIONS

We start with Eq. (37 of the text, which reads

$$y(t) = 2 \, \text{Re} \, \frac{ka}{2\pi} \, e^{j\sigma} \int_{-\frac{W''}{2}}^{\frac{W''}{2}} e^{j[c_3 \Omega + c_4 \Omega^2]} \, d\Omega \, . \qquad \text{(III-1)}$$

The exponent inside the integral can be written as

$$j\left[c_3 \Omega + c_4 \Omega^2\right] = j \, c_4 \left[\left(\Omega + \frac{c_3}{2 c_4}\right)^2 - \left(\frac{c_3}{2 c_4}\right)^2\right], \qquad \text{(III-2)}$$

so that (III-1) becomes

$$y(t) = 2 \, \text{Re} \, \frac{ka}{2\pi} \, e^{j\left(\sigma - \frac{c_3^2}{4 c_4}\right)} \int_{-\frac{W''}{2}}^{\frac{W''}{2}} e^{j c_4 \left(\Omega + \frac{c_3}{2 c_4}\right)^2} \, d\Omega \, . \qquad \text{(III-3)}$$

Let

$$\left.\begin{aligned} c_4 \left(\Omega + \frac{c_3}{2 c_4}\right)^2 &= \frac{\pi \alpha^2}{2} \\[2mm] \alpha &= \sqrt{\frac{2 c_4}{\pi}} \left(\Omega + \frac{c_3}{2 c_4}\right) \\[2mm] d\Omega &= \sqrt{\frac{\pi}{2 c_4}} \, d\alpha \end{aligned}\right\} . \qquad \text{(III-4)}$$

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Thus, we obtain

$$y(t) = \frac{ka}{\pi} \, \text{Re} \, e^{j\left(\sigma - \frac{c_3^2}{4c_4}\right)} \sqrt{\frac{\pi}{2c_4}} \int_{u_2}^{u_1} e^{j\pi \frac{\alpha^2}{2}} \, d\Omega \, . \tag{III-5}$$

This can be written as

$$y(t) = \frac{ka}{\sqrt{2\pi}} \, \text{Re} \, \frac{e^{j\left(\sigma - \frac{c_3^2}{4c_4}\right)}}{\sqrt{c_4}} \, [Z(u_1) - Z(u_2)] \, , \tag{III-6}$$

where $Z(u_1)$ and $Z(u_2)$ are the complex Fresnel integrals, with u_1 and u_2 given by

$$u_1 = \sqrt{\frac{2c_4}{\pi}} \left(\frac{c_3}{2c_4} + \frac{W''}{2}\right) \, ,$$

$$u_2 = \sqrt{\frac{2c_4}{\pi}} \left(\frac{c_3}{2c_4} - \frac{W''}{2}\right) \, . \tag{III-7}$$

We must take the real part of the product of several complex quantities. For convenience, let us list all the other symbols appearing in (III-6) in terms of the more elementary parameters; i.e.,

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$$a \quad = \frac{c+V}{c-V} \ , \quad \alpha = \frac{2V}{c-V} \ , \quad \lambda \ = \frac{2c}{c-V} \ ;$$

$$\sigma \quad = (t' + \gamma\, c_1)\, \omega_0'' + \gamma\, \lambda\, c_2\, (\omega_0'')^2 \ ,$$

$$c_3 \quad = t' + \gamma\, c_1 + 2\, \gamma\, \lambda\, c_2\, \omega_0'' \ ,$$

$$c_4 \quad = \gamma\, \lambda\, c_2 \ ,$$

$$c_1 \quad = \begin{cases} \dfrac{T}{W}\left(\omega_0 - \dfrac{W}{2}\right) & \text{for FM up} \ , \\[4mm] -\dfrac{T}{W}\left(\omega_0 + \dfrac{W}{2}\right) & \text{for FM down} \ , \end{cases}$$

$$c_2 \quad = \begin{cases} -\dfrac{T}{2W} & \text{for FM up} \ , \\[4mm] +\dfrac{T}{2W} & \text{for FM down} \ , \end{cases}$$

$$\omega_0'' = \omega_0 \left(\frac{c}{c+V}\right) - \frac{W}{2}\left(\frac{V}{c+V}\right) \ ,$$

$$\frac{W''}{2} = \frac{W}{2}\left(\frac{c}{c+V}\right) - \omega_0\left(\frac{V}{c+V}\right) \ ,$$

$$t' \quad = t - \frac{2R}{c} - T \ .$$

(III-8)

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We note that $\sqrt{c_4}$ is imaginary for FM up and real for FM down; i.e.,

$$\sqrt{c_4} = \begin{cases} j\sqrt{\gamma\lambda\dfrac{T}{2W}} & \text{for FM up} \\[3em] \sqrt{\gamma\lambda\dfrac{T}{2W}} & \text{for FM down} \end{cases} \qquad\qquad \text{(III-9)}$$

where $j = \sqrt{-1}$. Now let

$$\begin{bmatrix} x_0 = \dfrac{ka}{\sqrt{2\pi}} \;, \\[2em] x_1 = \sigma - \dfrac{c_3^2}{4c_4} \\[2em] \quad\; = -\dfrac{(t')^2 + 2\gamma\lambda c_1 t'}{4\gamma\lambda c_2} \;, \\[2em] x_2 = \sqrt{\dfrac{\gamma\lambda T}{2W}} \;, \\[2em] x_3 = \sqrt{\dfrac{\gamma\lambda T}{\pi W}}\left[\dfrac{c_3}{2\gamma\lambda c_2} + \dfrac{W''}{2}\right] \;, \\[2em] x_4 = \sqrt{\dfrac{\gamma\lambda T}{\pi W}}\left[\dfrac{c_3}{2\gamma\lambda c_2} - \dfrac{W''}{2}\right] . \end{bmatrix} \qquad \text{(III-10)}$$

Using these symbols, we have, for FM up,

$$y(t) = x_0 \, \mathrm{Re}\left\{\dfrac{\cos x_1 + j\sin x_1}{jx_2}\,[Z(jx_3) - Z(jx_4)]\right\} . \qquad \text{(III-11)}$$

Similarly, we obtain, for FM down,

$$y(t) = x_0 \, \mathrm{Re} \left\{ \frac{\cos x_1 + j \sin x_1}{x_2} \left[Z(x_3) - Z(x_4) \right] \right\} . \qquad \text{(III-12)}$$

All the x's are now real quantities. The complex Fresnel integral $Z(x)$ can be expressed (for real x) as follows:

$$\left. \begin{aligned} Z(x) &= C(x) + j \, S(x) \\ Z(jx) &= S(x) + j \, C(x) \end{aligned} \right\}, \qquad \text{(III-13)}$$

where

$$C(x) = \int_0^x \cos \pi \, \frac{\alpha^2}{2} \, d\alpha \; ; \; S(x) = \int_0^x \sin \pi \, \frac{\alpha^2}{2} \, d\alpha \; . \qquad \text{(III-14)}$$

These are tabulated functions. Using (III-13), we obtain

$$y(t) = \frac{x_0}{x_2} \left\{ [C(x_3) - C(x_4)] \cos x_1 + [S(x_3) - S(x_4)] \sin x_1 \right\} \qquad \text{(III-15)}$$

for FM up, and

$$y(t) = \frac{x_0}{x_2} \left\{ [C(x_3) - C(x_4)] \cos x_1 - [S(x_3) - S(x_4)] \sin x_1 \right\} . \qquad \text{(III-16)}$$

for FM down. (Note that x_1, x_3 and x_4 have different values for FM up and FM down.) We can express $y(t)$ in the following form:

$$y(t) = \left(\frac{x_0}{x_2} \right) R \cos (x_1 - \phi) \; . \qquad \text{(III-17)}$$

where

$$R = \sqrt{\left[C(x_3) - C(x_4)\right]^2 + \left[S(x_3) - S(x_4)\right]^2} \tag{III-18}$$

for both FM up and FM down, and

$$\phi = \begin{cases} + \tan^{-1}\left[\dfrac{S(x_3) - S(x_4)}{C(x_3) - C(x_4)}\right] & \text{for FM up}, \\[2em] - \tan^{-1}\left[\dfrac{S(x_3) - S(x_4)}{C(x_3) - C(x_4)}\right] & \text{for FM down}. \end{cases} \tag{III-19}$$

These expressions are exact (within the limits of our approximations), and can be evaluated with the aid of available tables of Fresnel integrals. However, we shall try to obtain approximate closed form representations for R and ϕ in the vicinity of the peak of the envelope.

Let us first consider the envelope R. We wish to determine its maximum. From the analysis leading to Eq. (16) in the text, we expect this maximum to occur near $c_3 = 0$. Let us postulate that the peak actually occurs at that point. A <u>necessary</u> condition for this to be so is that

$$\frac{dR}{dt} = 0 \; ,$$

when

$$c_3 = 0 = t' + \gamma c_1 + 2\gamma\lambda c_2 \omega_0''$$

$$= t - \frac{2R}{c} - T + \gamma c_1 + 2\gamma\lambda c_2 \omega_0'' \tag{III-20}$$

Let us check whether this condition is satisfied. From (III-18) we have,

$$\frac{dR}{dt} = \frac{1}{2R} \left\{ 2[C(x_3) - C(x_4)] \frac{d}{dt} [C(x_3) - C(x_4)] \right.$$

$$\left. + 2[S(x_3) - S(x_4)] \frac{d}{dt} [S(x_3) - S(x_4)] \right\} \quad . \tag{III-21}$$

Equation (III-21) will be equal to zero if <u>both</u>

$$\frac{d}{dt} [C(x_3) - C(x_4)] = 0, \text{ and}$$

$$\frac{d}{dt} [S(x_3) - S(x_4)] = 0, \tag{III-22}$$

provided that R does not equal zero at the same time. The Fresnel integrals in (III-22) are all of the form

$$\int_0^{x(t)} f(\alpha) \, d\alpha = F[x(t)] - F(0) \quad , \tag{III-23}$$

where F is the indefinite integral of f. Differentiating (III-23) with respect to t, we have

$$\frac{d}{dt} \int_0^{x(t)} f(\alpha) \, d\alpha = f[x(t)] \frac{dx(t)}{dt} \quad . \tag{III-24}$$

In our case, $x(t)$ is equal to either x_3 or x_4. Referring to (III-10), we see that

$$\left. \begin{array}{l} x_3 = k_1 c_3 + k_2 \\ \\ x_4 = k_1 c_3 - k_2 \end{array} \right\}, \tag{III-25}$$

where k_1 and k_2 are constants, and c_3 is of the form $c_3 = t +$ another constant. Hence,

$$\frac{dx_3}{dt} = k_1 = \frac{dx_4}{dt} \quad . \tag{III-26}$$

Thus, the $\frac{dx(t)}{dt}$ in (III-24) is equal to k_1, a constant. Now, the function $f(\alpha)$ in our case is either equal to $\cos\left(\frac{\pi}{2} \alpha^2\right)$ or $\sin\left(\frac{\pi}{2} \alpha^2\right)$. Combining all this information, (III-22) becomes

$$\left.\begin{array}{l}\dfrac{d}{dt} [C(x_3) - C(x_4)] = k_1 \left[\cos\left(\dfrac{\pi}{2} x_3^2\right) - \cos\left(\dfrac{\pi}{2} x_4^2\right)\right] \\[3mm] \dfrac{d}{dt} [S(x_3 - S(x_4)] = k_1 \left[\sin\left(\dfrac{\pi}{2} x_3^2\right) - \sin\left(\dfrac{\pi}{2} x_4^2\right)\right] \end{array}\right\} \tag{III-27}$$

We wish to find out whether both these equations in (III-27) are identically zero when $c_3 = 0$. Substituting (III-25) into (III-27), we obtain first

$$\begin{array}{l}\dfrac{d}{dt} [C(x_3) - C(x_4)] = k_1 \left[\cos \dfrac{\pi}{2} \left(k_1^2 c_3^2 + k_2^2 + 2k_1 k_2 c_3\right)\right. \\[3mm] \left. \qquad\qquad - \cos \dfrac{\pi}{2} \left(k_1^2 c_3^2 + k_2^2 - 2k_1 k_2 c_3\right)\right] \end{array} \tag{III-28}$$

Let

$$\left.\begin{array}{l} x = \dfrac{\pi}{2} \left(k_1^2 c_3^2 + k_2^2\right) \\[3mm] y = \dfrac{\pi}{2} (2k_1 k_2 c_3) \end{array}\right\} \quad . \tag{III-29}$$

Making use of the trig-identity

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \ ,$$

Eq. (III-28) becomes

$$\frac{d}{dt} [C(x_3) - C(x_4)] = k_1 [-2 \sin x \sin y] \ . \tag{III-30}$$

Now, since y contains c_3 as a factor, we see that $y = 0$ when $c_3 = 0$. Hence, $\sin y = 0$ when $c_3 = 0$, and

$$\frac{d}{dt} [C(x_3) - C(x_4)] = 0, \text{ when } c_3 = 0 \ . \tag{III-31}$$

Similarly, using (III-29) again and the trig-identity $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$, we obtain, for the second equation in (III-27),

$$\frac{d}{dt} [S(x_3) - S(x_4)] = k_1 [2 \cos x \sin y] \ , \tag{III-32}$$

which again contains $\sin y$ as a factor, yielding

$$\frac{d}{dt} [S(x_3) - S(x_4)] = 0, \text{ when } c_3 = 0 \ . \tag{III-33}$$

Thus we see that (III-22) is satisfied. It can readliy be checked that R is not identically zero when c_3 equals zero. Hence, we conclude that

$$\frac{dR}{dt} = 0 \text{ when } c_3 = 0 \left(\text{i.e., when } t = \frac{2R}{c} + T - \gamma c_1 - 2\gamma\lambda c_2 \omega_0'' \right). \tag{III-34}$$

Let us, therefore, think of c_3 as a shifted time variable; i.e., let

$$t'' = c_3 = t - \frac{2R}{c} - T + \gamma c_1 - 2\gamma\lambda c_2 \omega_0'' \tag{III-35}$$

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Note that Eq. (III-34) is satisfied regardless of the target velocity V. When $V = 0$, $\gamma = 0$ $\left(\text{since } \gamma = \dfrac{2V}{c-V}\right)$, which serves as a check.

Let us, therefore, accept our postulate as being true; namely, that the envelope R has its peak when $t'' = 0$.

When $t'' = c_3 = 0$, the arguments x_3 and x_4 of the Fresnal integrals are (from III-10) equal to

$$x_3 = \sqrt{\frac{\gamma \lambda T}{\pi W}} \left(\frac{W''}{2}\right) ,$$

$$x_4 = \sqrt{\frac{\gamma \lambda T}{\pi W}} \left(-\frac{W''}{2}\right) .$$

(III-36)

From (III-8) we see that

$$\frac{W''}{2} = \frac{W}{2}\left(\frac{c}{c+V}\right) - \omega_0 \left(\frac{V}{c+V}\right) ,$$

which, for $V \ll c$, reduces approximately to

$$\frac{W''}{2} \approx \frac{W}{2} - \omega_0 \frac{V}{c} .$$

Similarly, we have

$$\gamma = \frac{2V}{c-V} \approx \frac{2V}{c} ,$$

$$\lambda = \frac{2V}{c-V} \approx 2 ,$$

so that, letting $x = x_3 = |x_4|$ when $c_3 = 0$, we have

$$\left. \begin{aligned} x &= \sqrt{\frac{4VT}{c\pi W}} \left[\frac{W}{2} - \omega_0 \frac{V}{c}\right] \\ &= \sqrt{\frac{V}{c}} \left[\sqrt{\frac{TW}{\pi}} - \left(\frac{2V}{c}\right)\omega_0 \sqrt{\frac{T}{\pi W}}\right] \end{aligned} \right\} .$$

(III-37)

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Letting $B = \dfrac{W}{2\pi}$ and $f_0 = \dfrac{\omega_0}{2\pi}$, this becomes

$$x = \sqrt{\frac{V}{c}}\left[\sqrt{2TB} - \left(\frac{2V}{c}\right)f_0\sqrt{\frac{2T}{B}}\right] .$$ (III-38)

For the particular radar of interest, f_0 is approximately 1300 megacycles, and there are two pulse-compression systems to be considered:

System No. 1:

$$T = 1 \text{ millisecond}, \quad B = 1 \text{ megacycle} ;$$

System No. 2:

$$T = 2 \text{ milliseconds}, \quad B = 5 \text{ megacycles} .$$

The range of radial velocities we consider here are $0 \le V \le 10^4$ m/sec. Let us check if x_3 has a maximum or a minimum in this region; i.e., using (III-38), we solve for

$$\frac{dx}{dV} = 0 = \frac{d}{dV}\left[a\ V^{1/2} - b\ V^{3/2}\right]$$

$$= \frac{a}{2\sqrt{V}} - \frac{3}{2}\ b\ \sqrt{V} = 0.$$

For $V \neq 0$, we obtain $a - 3bV = 0$, or

$$V = {}_{3b} = \frac{\sqrt{2}\sqrt{\dfrac{TB}{c}}}{3\left[\dfrac{2\sqrt{2}\ f_0}{c\sqrt{c}}\sqrt{\dfrac{T}{B}}\right]} ,$$

which simplifies to

$$V = \frac{Bc}{6f_0} .$$ (III-39)

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For System No. 1, this corresponds to $V = 3.85 \times 10^4$ m/sec, which falls outside our region of interest. For System No. 2, the value is even larger. Thus, since there are no local maxima or minima between $0 \leq V \leq 10^4$ m/sec, we can compute the extreme values of x by taking the end points. For $V = 0$, we simply get zero. When $V = 10^4$ m/sec, we obtain

$$x \approx 0.237 \text{ for System No. 1};$$

$$x \approx 0.745 \text{ for System No. 2}.$$

(III-40)

Asymptotic series expansions of the Fresnel integrals $C(x)$ and $S(x)$ exist both in ascending powers of x (for $x < 1$) and in descending powers of s (for $x > 1$). The former are useful for $x < .< 1$ and the latter for $x >> 1$, since a few terms of the series then give us a good approximation.

From the above estimates, we see that at the peak of the envelope the argument of the Fresnel integrals is close to one when $V = 10^4$ m/sec. Thus, asymptotic expansions appear to be unsuitable in the immediate vicinity of the peak for large velocities.

We could, of course, consider smaller velocities. Suppose we require that

$$x = 0.01 << 1,$$

(where $x = x_3 = |x_4|$ when $c_3 = 0$).

Using Eq. (III-10), we find that this corresponds to a radial velocity of approximately

$$V \approx 10.5 \text{ m/sec for System No. 1, and}$$

$$V \approx 1.5 \text{ m/sec for System No. 2}.$$

However, for such low velocities, the analysis of a simple Doppler-shifted signal (as done in Appendix I) is expected to be quite adequate. It does not seem worthwhile to go through the necessary approximations.

Let us consider whether we can infer anything about the nature of $y(t)$ as we move away from the peak of the envelope. Going back to (III-10) and substituting t'' for c_3, we have

$$\begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix} = \sqrt{\frac{\gamma \lambda T}{\pi W}} \left[\frac{t''}{2\gamma \lambda c_2} \begin{Bmatrix} + \\ - \end{Bmatrix} \frac{W''}{2} \right] . \tag{III-41}$$

(Here we imply that + goes with x_3, − with x_4)

Let us rewrite (III-41) as

$$\begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix} = k_1 \left[\frac{t''}{k_2} \begin{Bmatrix} + \\ - \end{Bmatrix} k_3 \right] . \tag{III-42}$$

We have previously (III-40) found that the product $k_1 k_3$ is of the order one when $t'' = 0$. We would now like to know how large t'' has to be such that $|x_3|$ or $|x_4|$ are of the order 100 (so that we might use asymptotic expansions for large x).

We let

$$\left| \frac{t''}{k_2} \right| = 100 \ k_3 . \tag{III-43}$$

Substituting for k_2 and k_3, we have

$$|t''| = 100 \left[\frac{W}{2} \left(\frac{c}{c+V} \right) - \omega_0 \left(\frac{V}{c+V} \right) \right] |2\gamma \lambda c_2| . \tag{III-44}$$

This is approximately

$$|t''| \approx 200 \left[\frac{W}{2} - \omega_0 \frac{V}{c} \right] \left[\left(\frac{2V}{c} \right) \frac{T}{W} \right] .$$

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Using $V = 10^4$ m/sec, we have

$$|t''| \approx 6\mu\text{sec for System No. 1};$$

$$|t''| \approx 12\mu\text{sec for System No. 2}.$$

(III-45)

The following asymptotic approximations* are expected to be quite good for large values of x (say, $x \geq 100$).

$$C(x) \approx \frac{1}{2} + \frac{\sin \pi x^2/2}{\pi x} \; ;$$

$$S(x) \approx \frac{1}{2} - \frac{\cos \pi x^2/2}{\pi x} \; .$$

(III-46)

Letting $\alpha = \pi x^2/2$ and using the approximations in (III-46), we have

$$C(x_3) - C(x_4) = \frac{1}{\pi}\left[\frac{\sin \alpha_3}{x_3} - \frac{\sin \alpha_4}{x_4}\right],$$

$$S(x_3) - C(x_4) = \frac{1}{\pi}\left[\frac{\cos \alpha_4}{x_4} - \frac{\cos \alpha_3}{x_3}\right].$$

(III-47)

(Although we are using equal signs, it is understood that these are only approximations.)

Substituting (III-47) into (III-18), we have

$$R = \frac{1}{\pi}\left[\frac{1}{x_3^2} + \frac{1}{x_4^2} - \frac{2}{x_3 x_4} (\sin \alpha_3 \sin \alpha_4 + \cos \alpha_3 \cos \alpha_4)\right]^{1/2}$$

$$= \frac{1}{\pi}\left[\frac{1}{x_3^2} + \frac{1}{x_4^2} - \frac{2}{x_3 x_4} \cos(\alpha_3 - \alpha_4)\right]^{1/2} .$$

(III-48)

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*c.f. Watson "Theory of Bessel Functions," p. 545.

Let us rewrite (III-42) as

$$\begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix} = A \begin{Bmatrix} \pm \\ \end{Bmatrix} \Delta \ , \tag{III-49}$$

where

$$|\Delta| < < |A| \ .$$

Thus, we have

$$\frac{1}{x_3} = \frac{1}{A + \Delta} = \frac{1}{A\left(1 + \frac{\Delta}{A}\right)} = \frac{1}{A}\left[1 - \frac{\Delta}{A} + \left(\frac{\Delta}{A}\right)^2 - \ldots \right],$$

$$\frac{1}{x_3} = \frac{1}{A}\left(1 - \frac{\Delta}{A}\right) \text{ (ignoring higher order terms).}$$

Similarly,

$$\frac{1}{x_4} = \frac{1}{A}\left(1 + \frac{\Delta}{A}\right) \text{ (ignoring higher order terms),}$$

$$\frac{1}{x_3^2} + \frac{1}{x_4^2} = \frac{1}{A^2}\left(1 - \frac{2\Delta}{A} + \frac{\Delta^2}{A^2}\right) + \frac{1}{A^2}\left(1 + \frac{2\Delta}{A} + \frac{\Delta^2}{A^2}\right) = \frac{2}{A^2} \ .$$

Furthermore, we have

$$\frac{2}{x_3 x_4} = 2\left(\frac{1}{A}\right)\left(1 - \frac{\Delta}{A}\right)\frac{1}{A}\left(1 + \frac{\Delta}{A}\right) = \frac{2}{A^2}\left(1 - \frac{\Delta^2}{A^2}\right)$$

$$\frac{2}{x_3 x_e} = \frac{2}{A^2} \text{ (again, ignoring higher order terms).}$$

Similarly,

$$\alpha_3 - \alpha_4 = \frac{\pi}{2}\left(x_3^2 - x_4^2\right)$$

$$= \frac{\pi}{2}\left[A^2 + 2A\Delta + \Delta^2 - (A^2 - 2A\Delta + \Delta^2)\right]$$

$$= \frac{\pi}{2}\left[4A\Delta\right]$$

$$= 2\pi A\Delta \ .$$

Substituting the above approximations into (III-48), we have

$$R = \frac{1}{\pi}\left[\frac{2}{A^2} - \frac{2}{A^2}\cos 2\pi A\Delta\right]^{1/2}$$

$$= \frac{\sqrt{2}}{\pi A}\left[1 - \cos 2\pi A\Delta\right]^{1/2} \quad . \qquad \text{(III-50)}$$

But $1 - \cos 2x = 2\sin^2 x$, so that (III-50) becomes

$$R \approx 2\Delta\left(\frac{\sin \pi A\Delta}{\pi A\Delta}\right) \ . \qquad \text{(III-51)}$$

Referring back to (III-17), the whole expression for the envelope E is

$$E = \frac{x_0}{x_2} \ 2\Delta \ \frac{\sin \pi A\Delta}{\pi A\Delta} \ .$$

Substituting our elementary parameters back, we finally obtain

$$E = a\left(\frac{2k}{\pi}\right)\left(\frac{W''}{2}\right)\left[\frac{\sin t''\left(\frac{W''}{2}\right)}{t''\left(\frac{W''}{2}\right)}\right] \ . \qquad \text{(III-52)}$$

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This is a good approximation for

$$|t''| \geq 100 \left(\frac{W''}{2}\right) |2\gamma\lambda c_2| = 100 \frac{W''}{2} \left(\frac{2V}{c-V}\right)\left(\frac{2c}{c-V}\right) \frac{T}{2W} \approx 100 \left(\frac{2V}{c}\right) T .$$

Let us now look at the phase angle ϕ in the same region. We had

$$\phi = \pm \tan^{-1} \left[\frac{S(x_3) - S(x_4)}{C(x_3) - C(x_4)}\right] \quad \begin{Bmatrix} + \text{ for FM up} \\ - \text{ for FM down} \end{Bmatrix} .$$

Let

$$\psi = \pm \tan \phi .$$
(III-53)

Using (III-47), we have

$$\psi = \frac{\dfrac{\cos \alpha_4}{x_4} - \dfrac{\cos \alpha_3}{x_3}}{\dfrac{\sin \alpha_3}{x_3} - \dfrac{\sin \alpha_4}{x_4}} .$$
(III-54)

We have, as before,

$$\frac{1}{x_3} = \frac{1}{A} \left[1 - \frac{\Delta}{A} + \left(\frac{\Delta}{A}\right)^2 - \left(\frac{\Delta}{A}\right)^3 + \ldots \right] ;$$

$$\frac{1}{x_4} = \frac{1}{A} \left[1 + \frac{\Delta}{A} + \left(\frac{\Delta}{A}\right)^2 + \left(\frac{\Delta}{A}\right)^3 + \ldots \right] .$$

With $\Delta \ll A$, let us ignore all terms except the first. Then we get

$$\psi = \frac{\cos \alpha_4 - \cos \alpha_3}{\sin \alpha_3 - \sin \alpha_4} .$$
(III-55)

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But

$$\alpha_3 = \frac{\pi}{2} (A + \Delta)^2 = \frac{\pi}{2} (A^2 + \Delta^2 + 2 A \Delta) \; ;$$

$$\alpha_4 = \frac{\pi}{2} (A - \Delta)^2 = \frac{\pi}{2} (A^2 + \Delta^2 - 2 A \Delta) \; .$$

Letting

$$x = \frac{\pi}{2} (A^2 + \Delta^2) \; ,$$

$$y = \frac{\pi}{2} (2 A \Delta) \; ,$$

we have

$$\psi \approx \frac{\cos (x - y) - \cos (x + y)}{\sin (x + y) - \sin (x - y)} \; . \tag{III-56}$$

Using some trig-identities, we obtain

$$\psi = \frac{2 \sin x \sin y}{2 \cos x \sin y} = \tan x \; .$$

But $\psi = \pm \tan \phi$, so that $\phi = \pm x$. Substituting for x, we finally have

$$\phi = \pm \frac{\gamma \lambda T}{2W} \left[\left(\frac{t''}{2 \gamma \lambda c_2} \right)^2 + \left(\frac{W''}{2} \right)^2 \right] \begin{Bmatrix} + \text{ for FM up} \\ - \text{ for FM down} \end{Bmatrix} . \tag{III-58}$$

The carrier term is (from III-17) equal to $\cos (x_1 - \phi)$. Letting $\theta = x_1 - \phi$ we obtain, after a fair amount of algebra,

$$\theta \approx \omega_0 {}''t'' + \frac{\gamma \lambda T}{2W} \left[\pm (\omega_0'')^2 \mp \frac{(W'')^2}{4} \right] , \tag{III-59}$$

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where the upper signs are for FM up, and the lower signs for FM down. Now, since $W'' << \omega_0''$ by about three orders of magnitude, suppose we ignore the second term in the brackets. Substituting for γ and λ, we have

$$\theta \approx \omega_0 ''t'' \pm \left(\frac{2V}{c-V}\right)\left(\frac{2c}{c-V}\right)\left(\frac{T}{2W}\right)(\omega_0'')^2 \quad \left\{ \begin{array}{l} + \text{ for FM up} \\ - \text{ for FM down} \end{array} \right\}. \qquad \text{(III-60)}$$

Again, this is for

$$|t''| \geq 100 \left(\frac{W''}{2}\right)\left(\frac{2V}{c-V}\right)\left(\frac{2c}{c-V}\right)\left(\frac{T}{2W}\right) \approx 100 \left(\frac{2v}{c}\right) T \ .$$

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	Directorate of Radar and Optics, ESD, L.G. Hanscom Field, Bedford, Mass.

13. ABSTRACT

Equations for the phase and envelope of the output signal from a linear filter, matched to the transmitted signal, are derived. The transmitted signal is assumed to have a flat band-limited amplitude spectrum and a linear group delay. The input to the "matched filter" is the radar echo returned from a moving target whose velocity is essentially constant during the illumination time. It is shown that the returned signal is related to the transmitted signal by a time dilation. The resulting expressions for the phase and envelope are functions which involve Fresnel integrals. Approximations for these expressions are worked out. They are shown to be similar in form to those which are obtained when the returned signal is assumed to be transmitted signal by a Doppler shift.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Radar Targets Pulse Compression Linear FM						

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